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AN IMPRECISE PROBABILITY MODEL IN REPEATED INTERACTION GAMES

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We introduce an imprecise probability model for general interaction games based on the Dirichlet family of multivariate probability distributions. In order to deal with the lack of evidence (ex-ante information), an ambiguity averse attitude of the players is incorporated into the payoff function. The existence of an ambiguity averse Nash equilibrium for the N – stage game is established as a classical result, composed of individual Nash equilibria for each separate round in the repeated game. The existence of a common belief about the probability distributions is an essential assumption for the analysis. In addition, further research directions are suggested.

Key words: game; interaction; imprecise probability; ambiguity aversion; Nash equilibrium

INTRODUCTION

Game theory is a mathematical approach to interactive decision-making. To participate in a game means to investigate and act, separately or jointly, to achieve the most favourable outcome, typically in uncertain and complex environments. Each player, acting in accordance to some rules and information structure, tries to maximize the personal gain expressed by some payoff function. Whether best options will be sought for depends upon numerous factors such as situation awareness, individual preferences, abilities, beliefs and their consistency, the level of confrontation etc. A successful game model should incorporate all relevant components into a single operational structure.

Games formalize interactions in many ways. In a population of players, there are different situations in which players' acts are intertwined. The characteristics of their interaction depend not only of the individual set of strategy and the payoff function, but also of the entire strategy profile of the population. Circumstances in which players are constrained to choosing a single action (or choose from a set of available actions) when interacting within a certain subset of players from the population, give rise to *local interaction games* [1]. A generalized class of interaction games was defined by Morris [2], of which local interaction games and other more specialized classes of games (such as games of incomplete information and random matching games), can be viewed as special cases. This generalization enables a unified approach in the analysis of the incorporated classes of games, as well as a new insight of the problems arising in different categories of models that can occur in these classes.

In an incomplete information game, players are uncertain about the environment that they are in. A player (or a type of players) is uncertain about the opposing player or type, which can be expressed by saying that each player is one of a large set of possible types, and the type profile for all players is drawn from some distribution. A player in a random matching game is uncertain about who the opponent is, while in local interaction games, the player faces a distribution over the actions of some nearby opponents. These three classes of games share the feature that each player (or player type) tends to choose an action that is a best response to a distribution of his opponents' actions [3].

In describing situations that reflect uncertainty about future events or outcomes, the classical probabilistic models as defined by Kolmogorov are the usual choice. These models incorporate values which precisely estimate the uncertainty involved through an appropriate additive probability measure, without losing any information in the process. Mutually disjunctive elementary events are defined with probabilities expressed precisely by a number from the interval [0,1], with a request for unique previsions and conditional probabilities (for the non-zero probability events) to be determined. But in real decision situations, there is only limited information about probability distributions, which associates decision analysis with large uncertainty. It is very often the case that the probabilities of relevant events are ambiguous and precise values are impossible to define. Hence, the fulfillment of the requirements from the classical probability theory that arise from the additivity axiom is difficult and can lead to unavoidable measuring mistakes, incomplete statistical data, conflicting evidence and other problems, especially in situations where the factor of subjectivity is strong. As a result, the strict and highly demanding conditions of the classical probability calculus can not be considered appropriate for describing the problem when based on limited information.

To alleviate these difficulties, the precision requests concerning probabilities can be weakened to allow imprecise expression of individual probabilities. Imprecise probability is an extension and generalization of precise probability. Many concepts of precise probability theory can appropriately be generalized to imprecise probabilities, which express ‘uncertainty about the uncertainty’ or uncertainty of second order. These concepts allow overcoming the weaknesses of the traditional statistical approach and the often unmotivated assumptions for the applied functional forms of probability distributions. The imprecise probability models are needed in the inference processes when information is rare, unclear or conflicted, such is the situation in many real problems. In this sense, they can be considered as an essential step towards realistic decision making.

Game theory deals with the problem of uncertainty through applying the formal framework of Bayesian games. Still, its practical application is limited by the fact that quite often, the uncertainty is too complex to be adequately described by a classical, precise probability distribution. Simply applying tools from classical game theory to situations of complex uncertainty with only partial information about the current states of nature, could easily lead to wrong conclusions.

This paper presents a model for a particular class of games that are known as *interaction games*, as defined by Morris [2], with additional considerations for the element of uncertainty about the states of nature that are expressed through a Bayesian imprecise probability model. The Bayesian approach usually assumes a prior Dirichlet distribution on the state space and makes inferences by conditioning the prior distribution to the observed data. One of the reasons for assuming a Dirichlet distribution is its computational simplicity, due to the fact that it is a conjugate density function to the multinomial distribution. Therefore, the posterior density will be also Dirichlet, with parameters updated according to the observations.

In addition, ambiguity aversion of players is assumed. An event is ambiguous when the player does not know its probability. Ambiguity aversion is simply a preference of the known over the unknown. Player’s ambiguity attitude can be described upon defining a set of probability distributions over the set of all possible outcomes (states of the world). Then, ambiguity neutrality is expressed by indifference between all distribution mixtures in this set. Ambiguity aversion is exhibited if the player strictly prefers to restrict over some subset of these distributions. Conversely, the player exhibits a liking of ambiguity if strictly prefers the original distributions to some mixture over them.

The presented model is based on a repeated game structure, where a finite stage game is played multiple times (more precisely, there is a finite repetition of the same stage game). A usual convention holds - players know what all other agents did in the previous iterations, but have no knowledge of their moves in the current iteration. In this sense, it is an imperfect information game with perfect recall. The difference from classical models here is that the set of players with whom the interaction takes place, is not the same in each of the rounds. This issue is expressed by the payoff function, which is being adjusted according to the “evidence” i.e. the previous play.

The paper is organized as follows. Section 2 briefly describes the Dirichlet models of inference, presenting both the precise (PDM) and the imprecise (IDM) probability approach. The general interaction games of Morris are presented in Section 3. In Section 4, we present the model of Imprecise Probability Interaction Games (IPIG), by upgrading the interaction games with the elements from IDM. Some basic definitions are given and certain important issues are discussed. Finally, in Section 5 we conclude by discussing possible directions for further research.

THE IMPRECISE DIRICHLET MODEL IN A MULTIVALUE SAMPLING PROCESS

The power of the traditional probability theory to represent epistemic uncertainty has certain limitations. For example, the distributions that are used in probability models cannot recognize the situation of complete ignorance, when there is a total lack of information about the studied object or system. Such situations are usually described by applying uniform distributions, expecting that they will be further updated according to the new evidence. Still, the fact is that by introducing any form of distribution into the model, an extra knowledge has been added into the narrative.

Imprecision can result from many different circumstances, such as:

- not having enough data to determine a single prior belief;
- not being certain about the observation and measurement of data;
- the data are not specific (for example, when the observed data set is an unknown part of a bigger subset of the total state space);
- having several different opinions i.e. conflict or imprecision of the expertise;
- outliers or errors occurring in statistical sampling models, etc.

In this situations, imprecise probabilities (initial work by Walley, Fine, Kuznetsov) are defined as models for behaviour under uncertainty that correspond, in general, to a set of probability distributions. The theory that deals with imprecise probabilities is completely based on classical probability. As a generic term, imprecise probability refers to all mathematical models, both qualitative (imprecise) and quantitative (non-additive), that are not using sharp numerical measures for probability.

There are many ways in which imprecision can be expressed. Among them are probability intervals, sets of probability measures, lower and upper previsions, credal sets, belief functions, convex capacities, fuzzy measures. Although it might seem differently, all these models can be expressed in an equivalent manner by using lower and upper previsions. Thus, the theory of lower and upper previsions, introduced by Walley [4], provides the most general framework for incorporating imprecision in the models of decision-making.

The lack of information can be overcome by applying the Imprecise Dirichlet Model (IDM). Instead of a single density, IDM considers a set of prior densities on the parameter space. Having a set of densities instead of only one, the mathematical expectation for a measurable and bounded function with respect to all densities from this set will not be a single value, but a pair of lower an upper provisions, obtained by considering infimum and supremum values over the prior's set. It should be noted that the IDM model allows coverage not only of lack of evidence but is also suitable for cases where conflicting information from different sources exists [5].

First, let's investigate the Precise Dirichlet Model (PDM). The use of Dirichlet distribution in probability updating models is very appropriate because, next to the fact that the set of these distributions is very rich, any prior distribution can be approximated by a finite mixture of Dirichlet distributions. An important statistical property of PDM is that the density functions constitute a conjugate family with respect to multinomial likelihoods: if the prior is Dirichlet, the posterior distribution will also be a Dirichlet probability distribution.

To formulate PDM, we observe N realizations of the m possible states ω_i from the state space $\Omega = \{\omega_1, \dots, \omega_m\}$ according to the standard multinomial model. The probabilities for occurrence of each of the states from Ω are formalized as follows:

$$P(\omega_i) = \theta_i,\ \theta_i \geq 0 \quad \text{and} \quad \sum_{i=1}^{m} \theta_i = 1.$$

By defining n_i to be the number of observations of the state ω_i in N trials, we can construct a vector of random variables $n = (n_1, \dots, n_m)$, $\sum_{j=1}^{m} n_j = N$. In the PDM with parameters s and $(t_1, \dots, t_m)$, the prior probability distribution for $\theta = (\theta_1, \dots, \theta_m)$ is given by the density function:

$$p(\theta) = \Gamma(s) \cdot \frac{\prod_{i=1}^{m} \theta_i^{st_i - 1}}{\prod_{i=1}^{m} \Gamma(st_i)},$$

where Γ is the Gamma-function. The posterior distribution is given by the density function:

$$p(\theta | n) \propto \prod_{i=1}^{m} \theta_i^{n_i + st_i - 1},$$

which is also a Dirichlet distribution when multiplied by the multinomial likelihood function relative to n, with updated parameters $N + s$ and $t^* = (t_1^*, \dots, t_m^*)$, where $t_i^* = (n_i + st_i)/(N + s)$.

The hyper-parameter $s > 0$ in the PDM determines the influence of the prior over the posterior distribution: the bigger its value, the greater is the uncertainty about the observations and consequently, the convergence of the upper and lower probabilities will be slower, and the conclusions should be more cautious. The value of this parameter should not depend on the number m of all possible states of nature or the total number N of observations. Walley [4] has defined *s* as the number of observations that will reduce the difference between upper and lower probabilities to half of its initial value. Smaller values of *s* will signify faster convergence with increased precision of the conclusions, while for large values of *s* the conclusions will be weaker. Some authors adopt different value ranges for *s*, for example: $s > 0$, $s > 1$, $s \in [1,2]$ etc.

Each of the t_i is the mean value of the respective θ_i. In order to reliably choose a fixed value for t_i, the experimental evidence should be extensive i.e. the value of *N* should be high. There are many situations where this is not possible. The alternative approach is to switch to the Imprecise Dirichlet Model (IDM), which takes into consideration all possible values for $t_i \in (0,1)$ i.e. the entire interior of the *m* - dimensional unit simplex Δ^m.

Now, if *n(A)* is the number of observations of the subset *A* of Ω, then the predictive probability *P(A,s)* of *A* with the Dirichlet prior from PDM, relative to *N*, will be given as:

$$P(A \mid s,t,n) = \frac{n(A) + s \cdot t(A)}{N+s}, \text{ where}$$

$$t(A) = \sum_{\omega_i \in A} t_i .$$

This probability value can be maximized and minimized over $(t_1,\ldots,t_m) \in \Delta^m$, in order to obtain the posterior lower and upper predictive probabilities of *A*:

$$\underline{P}(A \mid n,s) = \frac{n(A)}{N+s},$$

$$\overline{P}(A \mid n,s) = \frac{n(A)+s}{N+s}$$

Hence, the probability for occurrence of some element (state) from *A*, will be a number from the interval between these two values. If it is not certain which of the states $\omega_i \in A$ have been observed, the lower and the upper bounds for the probability of *A* can still be estimated, taking into account all possible $k = 1, \ldots, M$ subsets of the state space Ω and all $(t_1,\ldots,t_m) \in Int\Delta^m$ from the interior of the *m* – dimensional unit simplex Δ^m:

$$\underline{P}(A,s) = \min_k \inf_t \frac{n^{(k)}(A) + s \cdot t(A)}{N+s},$$

$$\overline{P}(A,s) = \max_k \sup_t \frac{n^{(k)}(A) + s \cdot t(A)}{N+s}.$$

Clearly, $n^{(k)}(A) = \sum_{\omega_i \in A} n_i^{(k)}$, where $n^{(k)}(A)$ is the number of observations of the enumerated combination of states *k*. The infimum value of $t(A)$ is 0 and the supremum is 1 for all $A \neq \Omega$, while for $A = \Omega$ we have a unique value $t(A) = 1$.

For considering the minimum and the maximum of $n^{(k)}(A)$, we divide the power set of Ω in three parts: the family F_1 of subsets of A, the family F_2 of sets *B* such that $B \cap A = \varnothing$ and the family F_3 of sets that do not belong to $F_1 \cup F_2$. Then, setting c_i to be the number of occurrences of the set A_i, we have:

$$\min_{k=1,\ldots,M} n^{(k)}(A) = \sum_{A_i \in F_1} c_i, \text{ and}$$

$$\max_{k=1,\ldots,M} n^{(k)}(A) = N - \sum_{A_i \in F_2} c_i = \sum_{A_i \cap A \neq \varnothing} c_i .$$

Choosing an interval in order to present imprecision of knowledge or observations is suitable for several reasons. Statistical distributions need assumptions of distribution types, distribution parameters and a mapping from events to real values between 0 and 1. Fuzzy sets need assumptions of not only lower and upper bounds, but also membership functions. The interval on the other hand is presented in a simple form, by a pair of numbers (the lower and the upper bound), is easily understandable and does not assume any kind of distribution. Given that the essence of epistemic uncertainty is the lack of knowledge, a representation with the least assumption is the most desirable.

INTERACTION GAMES

When a large population of players interacts strategically, some encounters may be more likely to happen than others. From a game – theoretical perspective, this situation becomes interesting when it is assumed that the player cannot decide separately for each possible encounter (group of interacting players), but instead must choose a fixed strategy that will be played against all of them. This situation should not be confused for incomplete information in games, which is a canonical way of modelling strategic environments in the presence of uncertainty about players' preferences, their beliefs about other player' beliefs about preferences, and so on. In interaction games, large populations of players interact strategically without uncertainty, but dealing only with subsets of the total population.

The generalized interaction games have been defined by Morris [3]. They include a finite or a countably infinite population of players, each of which participates in a game with a random group (a subset) of other players. The convention in game theory is that the individual payoffs of all players depend on the strategic profile of the entire population – in this case, of the group of participating players. Interaction games have two more conventions:

(i) in each of the repeated encounters, the player should choose the same strategy, and

(ii) the final payoff will not depend on the strategies of the non-participating players.

Interactions have weights, and the equilibrium of the interaction game is a strategic profile that enables each player to maximize the weighted sum of payoffs from each interaction. The definition of the equilibrium in general interaction games corresponds to the standard Nash Equilibrium in games with incomplete information [2].

Below we briefly present the formal model of general interaction games.

A population X of players is observed, which is finite or possibly countably infinite. For each player $x \in \boldsymbol{X}$, a set A_x of actions (pure strategies) a_x of x for the standard strategic form game is given, and a payoff function $u_x : A \to \boldsymbol{R}$ is defined over the product space $A = \underset{x \in \boldsymbol{X}}{\otimes} A_x$. The individual mixed strategies α_x of x are defined by probability distributions over A_x i.e. $\alpha_x \in \Delta A_x$. If the strategic set A_x is infinite, then it is agreed that the mixed strategies should have a finite support. Mixed strategy profiles that include all players can be represented by vectors $\alpha \equiv \{\alpha_x\}_{x \in X}$.

In the interaction game, each player can be involved with a random group of other players. The possible interaction groups are described by a set I of subsets from X (elements of the power set 2^X), such that the elements of an arbitrary set from I are the participating players in a single (one shot) game. The elements of I are called *interactions*. In addition, I_x denotes the set of interactions of the player $x \in X$,

$$I_x = \{Q \mid Q \in I,\ x \in Q\}.$$

Clearly, $I = \bigcup_{x \in X} I_x$. It can be agreed that the cardinality of all elements of I is not less than 2, in order to exclude the degenerate cases of the 'zero player' games and games with only one player, which is a classical case of decision-making. Further on, we will denote pure and mixed strategies that are played within an interaction Q by (a, Q) and (α, Q) respectively.

The likelihood of a particular interaction to effectuate is defined by a weight function $P : I \to \boldsymbol{R}^+$, such that for all $x \in X$ it holds that $0 < \sum_{Q \in I_X} P(Q) < \infty$. The last condition ensures that each player will participate in at least one interaction, but also that the total participation of the player in different interactions will be bounded.

Now, the pure strategy payoff for the player x can be defined as a weighted sum of the payoffs u_x from the individual interactions, depending on the chosen pure strategy a:

$$v_x(a) : A = \underset{x \in \boldsymbol{X}}{\otimes} A_x \to \boldsymbol{R}, \text{ defined by}$$

$$v_x(a) = \sum_{Q \in I_X} P(Q)\, u_x(a, Q).$$

For the latest sum to be well defined, it is assumed that the payoffs $u_x(a, Q)$ are bounded for all x. Similarly, the payoff that x receives from a mixed strategies profile α that was played in the interaction Q, can be defined as:

$$v_X(\alpha) = \sum_{Q \in I_x} P(Q)\, u_X(\alpha, Q), \text{ where}$$

$$u_X(\alpha_Q, Q) = \sum_{a_x} \left(\prod_{y \in X} \alpha_y(a_y) \right) u_X(\alpha, \mathrm{Q}).$$

The general class of interaction games allows a unified representation of several other classes (such as incomplete information, local interaction and random matching games), by capturing their common structural elements. A dynamic interpretation of the model with continuum of players, has also been formulated and discussed [3]. The common structure of interaction games helps in better understanding each of the separate classes of games.

THE PROPOSED MODEL

In this section, we define the imprecise probability model of a repeated interaction game, by introducing Walley's formulation into the general class of interaction games as defined by Morris. A *repeated game* is an extensive form game that consists of a number of repetitions of some base game, called *stage game*. Thus, the stage game is a normal (one shot) game in which players act simultaneously. In the proposed model, each player chooses a strategy, while repeatedly faced with uncertainty about the set of participating players (opponents) that are involved in the interaction. The uncertainty comes from not having ex-ante information about the group of players (a subset of the total set of players that participate in the game) that will interact in the next stage. In response to this situation, the states of nature are presented by the interaction subsets of players and the IDM model is applied to express the uncertainty.

Definition 1. The model of an Imprecise Probability Interaction Game (IPIG) consists of:

i. a set of players X;

ii. a set of pure strategy profiles $A = \underset{x \in X}{\otimes} A_x$;

iii. a payoff function $v = \{v_x\}_{x \in X}$ defined on A;

iv. a finite set of interactions $I = \{Q_j \mid j = 1, ..., m\}$ (the state space);

v. an imprecision parameter s.

In addition, we make some structural assumptions and clarifications.

A1. Each player x interacts within sets of players. The set of all interactions of x is denoted by I_x and it is a subset of I.

A2. The sets X, A and I and the payoff v are common knowledge.

A3. The players are not sure about the interaction they will be involved in.

A4. The total payoff received by a player x is the sum of the payoffs from each of the previously participated interactions.

A5. At each stage, only one interaction from I can take place.

With the above settings, the IPIG model is a tuple (X, A, v, I, s) representing an upgrade of the general interaction game that incorporates uncertainty. The individual mixed strategies are defined in the usual way, as are the combined profiles of pure and mixed strategies.

Let's suppose that there are total of m interaction groups, $card(I) = m$, and that the elements (interactions) Q_j (j = 1, ..., m) from I are appropriately indexed. Since the probability of a player x to observe a particular interaction $Q \in I_x$ is imprecise, the payoff function cannot be deterministically formulated. Instead, we will define lower and upper limits of the expected payoff, using the previously discussed IDM.

Definition 2. For a mixed strategy α, the total expected payoff of player x with respect to the probability distributions over the interaction's set I is the weighted sum of the payoffs from all interactions of x when α is played, integrated over the distribution space Δ^m:

$$Ev_x(\alpha) = \int_{\pi \in \Delta^m} \sum_{j=1}^{m} \left(v_x(\alpha, Q_j) \cdot \pi_j \right) p(\pi)\, d\pi .$$

Here π has the precise multinomial Dirichlet distribution over I, π_j is the probability of Q_j according to π, and $v_x(\alpha, Q_j)$ denotes the payoff of x from an engagement in the interaction Q_j when the strategy profile α was observed.

The subjective probability p over the unit simplex Δ_I (a second-order probability over I) expresses the ambiguous attitude of the player, i.e. the subjective uncertainty about the "true" probability π. We have:

$$Ev_x(\alpha) = \sum_{j=1}^{m} v_x(\alpha, Q_j) \int_{\pi \in \Delta^m} \pi_j\, p(\pi)\, d\pi = \sum_{j=1}^{m} v_x(\alpha, Q_j)\, E_p \pi_j ,$$

where $E_p \pi_j$ is the expected value of π_j in respect to the probability p that the particular distribution π, of which π_j is a component, will be observed.

According to the pervious discussions related to the lack of evidence, the expected value for the probability π_j for observing Q_j with respect of $p(\pi)$ can be estimated by $E_p \pi_j = \frac{n_j + st_j}{N + s}$, which leads to the following expression:

$$Ev_x(\alpha) = \sum_{j=1}^{m} v_x(\alpha, Q_j) \cdot \frac{n_j + st_j}{N + s} . \quad (1)$$

The ambiguity about the parameters that are applied in (1) imposes payoff concerns and motivates an ambiguity averse approach in search for a possible strategic advantage, when faced with the uncertain interaction set. As before, for the purpose of eliminating the hyper-parameter t, we introduce intervals for the expected payoff $v_x(\alpha)$ by accounting all values of $\left(t_1,\ldots,t_m\right)\in\Delta^m$ in (1):

$$\underline{E}\,v_x(\alpha)=\inf_{t\in\Delta^m} Ev_x(\alpha)\,,$$

$$\overline{E}\,v_x(\alpha)=\sup_{t\in\Delta^m} Ev_x(\alpha)\,.$$

Consequently, the payoff function v_x of x will be presented as an interval:

$$\left[\underline{E}v_x(\alpha),\overline{E}v_x(\alpha)\right].$$

The interval-valued expected payoffs can be compared according to different criteria. Among them are: the criterion of maximality, the concept of admissibility, the ambiguity aversion approach etc. In order to define a solution concept for a game, a specific criterion should be chosen. Here, we refer to the pessimistic (lower limit) payoff evaluation and define the strict ambiguity aversion under the previously described circumstances. Ambiguity averse behavior is often viewed as a robust response to doubts about beliefs. Under strict ambiguity aversion every strategy is evaluated by its minimal expected payoff, allowing the interval-valued expectations to be replaced by the correspondent lower interval limits.

In defining the expected ambiguity averse payoff for a given mixed strategy, we follow a representation approach given in [7].

Definition 3. The player x's payoff under a mixed strategy profile α and strict ambiguity aversion in a repeated game is defined as:

$$V_{\alpha,x}\;=\;\frac{s}{N+s}\cdot V_0\;+\;\sum_{k=1}^{m}\frac{c_k}{N\;+s}\cdot V_k\,,\qquad(2)$$

where N is the number of stage games in the repeated game, c_k is the number of occurrences of the interaction Q_k, $V_i=\min_{j\in\Delta^m} v_x\left(\alpha,Q_j\right)$ for $i=1,\ldots,n$, and $V_0=\min_{j=1,\ldots,m} v_x\left(\alpha,Q_j\right)$.

This definition takes into account all payoffs from previous stage games, thus it is history-related and involves an element of learning.

Next, we define a Nash equilibrium solution concept for a repeated imprecise interaction game under strict ambiguity aversion. This definition implicitly involves an updating assumption that takes into consideration the observed play and the received payoffs from the previous stage games. It should be noted here that the described situation differs from the standard repeated game model, where the stage game is always the same and consequently, the circumstances supporting a Nash equilibrium play remain invariable. In our model, the situation changes with the history since interactions may vary, with the results from these changing interactions of the previous rounds being incorporated into the payoff function.

Definition 4. The strategy profile $\alpha^*\in\Delta A$ is *an ambiguity averse Nash equilibrium* of a stage game in the imprecise probability interaction game (X, A, v, I, s), if for all $x\in X$ and all mixed strategy profiles $\alpha\in\Delta A$, it holds that:

$$V_{\alpha^*,\,x}\;\geq\;V_{(\alpha^*_{-x},\alpha_x),\,x}\;.$$

Here, α^*_{-x} denotes the oponents' strategy profile for the player x within the strategy profile α^*, while α_x is an arbitrary strategy of x. The equilibrium definition requires that given the expected payoff (2), each of the players best responds to the strategies of all other players in the game. To the extent that there is a lack of evidence to precisely define the Dirichlet prior over the state space, the ambiguity related to its parameters translates into ambiguity about the equilibrium play. By applying ambiguity aversion to the parameter t (more precisely, a pessimistic approach by endorsing minimum payoff values) the payoff function (2) incorporates only one parameter - namely s, whose value in general may not be shared by all players. In case there is a common belief about the value of s, the defining solution of each separate stage game will coincide with a classical Nash equilibrium of a normal form game. We will formalize this discussion in the following theorem.

Theorem. Let G be a repeated game for the finite imprecise probability interaction game $\left(X,A,v,I,s\right)$ with finite population of players. If for the first N rounds in G, there is a common belief about the value of the parameter s of the Dirichlet prior over the set of all possible interactions, then there exists a (mixed) ambiguity averse Nash equilibrium for G in the N - stage game and it is a Nash equilibrium sequence of ambiguity averse Nash equilibriums from each of the first N rounds.

The idea underlying this theorem and its proof comes directly from the classical result for equilibria existence in (complete information) finite strategic form games. Having that any sequence of stage-game Nash equilibria is a subgame-perfect equilibrium (SPE) in a finite repeated-game (i.e. it is a Nash equilibrium in every subgame of the original game) [8], this theorem also ensures the existence of a SPE in this model of imprecise probability interaction games.

The existence of a common belief about the probability distributions is an essential assumption for the analysis. Depending on the beliefs that players hold for the unknown elements (probability distributions, parameters), ambiguity aversion may expand, shrink or simply change the equilibrium set. Holding payoffs and the structure of the underlying game fixed, ambiguity aversion may expand the set of equilibria relative to the groups that share a common belief about distributions. Without this restriction (i.e. taking into account all possible distributions), ambiguity aversion will not affect the equilibrium set [6].

As previously discussed, it is not necessary for the value of the parameter *s* to be identical for all players, as long as its individual estimations are common knowledge. Alternatively, if this is not the case i.e. players have private estimations for *s* that are not publicly known, a Bayesian game can be considered where a player could analyze the game conditioning on various players' types, where the prior probability distribution over types is assumed to be common knowledge. When appropriate, this can be presented by enlarging the interaction set *I* in the game model. The result would be a Bayesian (Nash) equilibrium (BNE), a straightforward extension of the Nash equilibrium which depicts the uncertainty about the parameters and the way in which players react to that uncertainty. In BNE, each type of player chooses a strategy that maximizes expected utility given the actions of all types of other players and their beliefs about other players' types.

FURTHER RESEARCH CONSIDERATIONS

In this paper, we have introduced an imprecise Dirichlet model for general interaction games. In order to deal with the lack of evidence (ex-ante information), an ambiguity averse attitude of the players is incorporated into the payoff function. This approach ensures the existence of a (sequential) Nash equilibrium for the N – stage game, composed of individual Nash equilibria for each separate round. In essence, being a combination of changing interactions, uncertainty and ambiguity, the model raises many research questions.

There are various directions in which the analysis of this game model can unfold, each referring to different issues and considerations. To begin with, ambiguity, unlike fundamental uncertainty, may disappear with the passage of time simply because the increasing evidence will provide means for improved estimation of the unknowns, in this case the parameters of the applied IDM. While considering new evidence, the strict ambiguity criterion that is applied to game payoffs can easily prove to be over-pessimistic. For this reason, more sophisticated representations of the interval-valued expected payoffs that consider players' attitude towards ambiguity are desirable.

Possible modifications of the model can account for different payoff functions that depart from the ambiguity – averse standpoint or are adjusted to an infinite repeated game model that incorporates a discount factor. Another obvious possibility is to consider a replacement of the imprecise Dirichlet model with a different type of uncertainty presentation such as, for example, belief functions of the Dempster – Shafer theory of evidence. The later is actually a generalization of probability theory which, by assigning probability to sets (of events) instead of singletons, enables a more abstract approach towards evidence at hand. Moreover, if the available evidence permits assignment of precise probabilities to single events, the Dempster – Shafer theory will seize down to the traditional probabilistic model.

The proposed model may be analyzed more in detail in reference to subfamilies of the generalized interaction games, such as random matching games or local interaction games. The influence of the parameter *s* on the equilibrium behavior could be another point of interest. Finally, complementing the assumption of ambiguity aversion with the assumption of dynamic consistency, which can lead to equilibrium sets of games with ambiguity averse players coinciding with the equilibrium sets of Bayesian games [9], can also be considered.

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МОДЕЛ НА НЕПРЕЦИЗНА ВЕРОЈАТНОСТ ВО ПОВТОРЕНИ ИГРИ НА ИНТЕРАКЦИЈА

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Воведуваме модел на непрецизна веројатност кај општите игри на интеракција, врз основа на фамилија мултиваријантни распределби на веројатност на Дирихле. Притоа, отсуството на претходна информација е искажано преку аверзија кон повеќезначност кај играчите, која е вградена во функцијата на добивка. Дефиниран е Нешов еквилибриум со аверзија кон повеќезначност во повторената игра на интеракција, кој се состои од Нешови еквилибриуми на секоја од рундите и утврдено е неговото постоење, врз основа на класичната теорија. Клучна претпоставка во анализата е општото верување за распределбите на веројатност во моделот на Дирихле. На крајот, дадени се насоки за идни истражувања.

Клучни зборови: игра; интеракција; непрецизна веројатност; аверзија кон повеќезначност; Нешов еквилибриум